Construction And Spectra Of Non-Regular Minimal Graphs

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Abstract

The number of distinct eigenvalues of the adjacency matrix of graph G is bounded below by d(G)+1, where d is the diameter of the graph. Graphs attaining this lower bound are known as minimal graphs. The spectrum of graph G, where G is a simple and undirected graph is the collection of different eigenvalues of the adjacency matrix with their multiplicities. In this paper, we consider the minimal graphs such as the Petersen graph, cycles of even length, complete bipartite, K_4,4 by deleting one factor and constructing a non-regular class of graphs. Determine characteristic polynomials and spectra of constructed graphs proving that the constructed class of nonregular graphs are minimal graphs.

Keywords:

Characteristic polynomial; Diameter of a graph; Divisor graph; Minimal graphs; Spectra of a graph.

1. Introduction

The theory of linear algebra, particularly the theory of matrices, is a useful tool for analyzing the structural aspects of graph spectra, and the properties of graphs can thus be examined using the spectrum of their adjacency matrix and the Laplacian matrix. Cvetkovic in 1971, gave the method of determining the number of trees using the spectral method [1]. Recent works [2-4] have studied the graphs with fixed eigenvalues. The main eigenvalues of graphs and signed graphs are studied by researchers [5-8]. The class of connected graphs with exactly k main eigenvalues for each positive integer k using a divisor graph of certain simple graphs is found in [9]. Researchers in a variety of sectors of science and design engineering, use graph spectra and their implementation to better understand the fundamental features of graphs. Spectral clustering is used to resolve architecture issues such as water supply network partitioning using graph spectral techniques and mathematical approaches for water distribution network management [10,11]. For the past decade or so, computer science professionals have studied graph spectra, and a literature review reveals a wide range of applications in the field [12]. Mathematically determining the eigenvalues of large matrices is a topic for which researchers are looking for developing improved computer algorithms [13].

Willem H. Haemer used extensively interlacing eigenvalues techniques and obtained various bounds for graph parameters [14,15] which serve as basic inequalities referred by many researchers. Bao-FengWu et. al., [16] studied interlacing eigenvalues on some operations of graphs. A divisor graph is a multi-digraph constructed from graph G, which is a robust method to establish the spectrum of the graph with a lesser size matrix. The divisor graph comprises the main part of the spectrum of the graph [17] (pp.116-153), [18] (pp.37-45). A graph having all vertices of the same degree is called a regular graph. In the literature, the spectrum of regular graphs is easily defined.

The different eigenvalues in the graph's adjacency matrix are bounded below by the graph's diameter plus one. Minimal graphs are those that achieve this lower bound. Regular graphs and strongly regular graphs are two types of minimal graphs that are well-known. Robert A. Beezer has worked on the direction and presented a recursive generation technique for creating graphs with the fewest eigenvalues possible [19]. The results here in this paper are a combination of the above-mentioned work's inspirations, as well as the notion of a divisor and the results presented in [20,21].

In this paper, we present the construction of a class of non-regular minimal graphs with large matrices by taking into account the existing minimal graphs. The Peterson graph, the complete bipartite graph, the even-length cycles, the graph obtained by deleting one factor from K_4,4 are considered in this study. We use the concept of a divisor graph to determine characteristic polynomials and find the spectra of the constructed class of graphs. And, prove that the construction of the new class of graphs results in non-regular minimal graphs.

1.1 Material and existing results

Graphs considered in this study are simple. For basic definitions, terminology, and notations, the reader can refer to [22]. The definitions used to understand the construction and proofs of the theorems presented in this paper are given here.

A multi-digraph is a directed graph that is allowed to have multiple arcs.

Adjacency matrix $\boldsymbol{A}$ of a multi-digraph G whose vertex set $\{\boldsymbol{X_1,X_2,\ldots,X_k}\}$ is a square matrix of order k, whose $\boldsymbol{(i,j)}^{th}$ entry is equal to the number of edges (arcs) starting from the vertex x_i and terminating at the vertex $\boldsymbol{x_j}$.

The spectrum of graph $\boldsymbol{G}$ is the collection of different eigenvalues of the adjacency matrix $\boldsymbol{A}$ with their multiplicities. The set of simple adjacency eigenvalues of the graph is said to constitute the main part of the spectrum of a graph [23].

Given an s×s matrix $\boldsymbol{B=(b_{ij})}$, let the vertex set of a graph G be partitioned into (non-empty) subsets X_1 , X_2 ,..., X_s so that for i,j=1,2,…,s each vertex from X_i is adjacent to exactly b_ij vertices of X_j. Then the multidigraph F with adjacency matrix $\boldsymbol{B}$ is called a front divisor of G, or briefly, a divisor of G [Book 18, pp.116-122].

Definition in [14], *Let* $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$ and $\mu_1 \geq \mu_2 \geq \ldots \geq \mu_m$ with $m < n$ be the two sequences of real numbers. Then sequence second interlace the first one if, $\lambda_i \geq \mu_i \geq \lambda_{n-m+i}$; $i=1,2,\ldots..,m$.

2. Results and discussion

In this section, we show the construction of a nonregular minimal graph using Petersen graphs for example. For all other graphs considered, the construction method can be followed.

2.1. Construction

Let P stands for the Petersen graph, C_6 for the cycle on 6 vertices, $K_{n,n}$ for the complete bipartite graph on $2n$ vertices and H for graph obtained by deleting one factor from $K_{4,4}$. Construct a graph G by joining one vertex of each of the l copies of the Petersen graph P's to a newly inserted isolated vertex v, *so* that the degree of the vertex v is l. Recognize the graph class $G(v,lP,1)$ in Fig 1, for convenience.

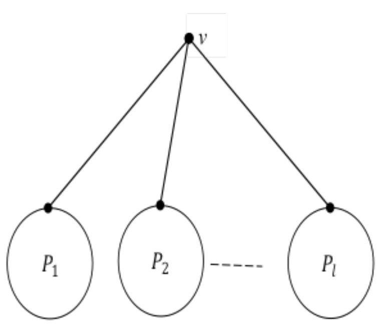

Fig. 1. Graph $G(v,lP,1)$

Similarly, we can create a class of graphs taking graphs C_6, $K_{n,n}$ and H, and label the class of graphs $G(v,lC_6,1)$, $G(v,lK_{n,n},1)$ and $G(v,lH,1)$.

2.2. Characteristic polynomials

The analysis of the characteristic polynomials of the class of minimal graphs generated in section 2.1, is the subject of this section. Before we start looking for the characteristic polynomials of the graph classes, we quote some important theorems.

2.2.1 Theorem [Book 18-pp. 19-20]:

Let A be a real symmetric matrix with eigenvalues $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_p$. Given a partition of the vertex set $\{1,2,\ldots,p\} = \Delta_1 \cup \Delta_2 \cup \ldots \cup \Delta_m$ with $|\Delta_i| = p_i > 0$. Here $|\Delta_i|$ denotes the cardinality of the set Δ_i. Consider the corresponding block $A=[A_{ij}]$ where A_{ij} is a $p_i \times p_j$ block matrix. Let blocks A_ij have constant row sums b_{ij}. Let $B=[B_{ij}]$ is a principal submatrix of the matrix A, then the spectrum of matrix B is contained in the spectrum of matrix A.

2.2.2. Theorem [14]:

The theorem states that if, B is a principal submatrix of a symmetric matrix A with corresponding adjacency eigenvalues $\mu_1 \geq \mu_2 \geq \ldots \geq \mu_m$ and $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$ respectively. Then the eigenvalues of B interlace the eigenvalues of A.

2.2.3. Theorem [18]:

The spectrum of any divisor graph contains the main part of the spectrum of the original graph.

2.2.4. Theorem [18]:

Any divisor of a graph G has the index-(maximum eigenvalue) of G as an eigenvalue.

2.2.5. Theorem [17]:

A graph G with diameter d has at least d+1 distinct eigenvalues.

2.2.6. Definition [14]:

Graphs having spectrum containing exactly d+1 different eigenvalues, where d is the diameter of the graph are called minimal graphs.

2.2.7. Lemma [17]:

Consider M a non-singular matrix with size $p \times q$, orders of matrices N is p×n, P is m×q and Q is m×n, we have a square matrix,

|■(M&N@P&Q)|=|M||Q-PM^(-1) N|

$$\begin{vmatrix} M & N \\ P & Q \end{vmatrix} = |M||Q - PM^{-1}N|$$

Now, we find the characteristic polynomials of the constructed graph classes, given as theorems.

Theorem 1: For vertex v and $l \geq 1$, the characteristic polynomial of $G(v,lP,1)$ is as given in the following equation,

$$\emptyset(G(v,lP,1):\lambda) = (\lambda-3)^{l-1}(\lambda+2)^{4l-1}(\lambda-1)^{5l-1}(\lambda^4-2\lambda^3 - (5+l)\lambda^2+(6+2l)\lambda+2l \quad ; \quad for \quad l \geq 1$$

Proof: Let for simplicity the graph $G(v,lP,1)$ be referred to as G. We observe that G and the vertex deleted graph G-v are graphs of order 10l+1 and 10l respectively.

Let A and B be the adjacency matrices of the graphs G and G-v respectively. The eigenvalues of G-v(*i.e.,* l copies of Petersen graph) are l times 3, 5l times 1, 4l times -2. Denote the eigenvalues of G-v as μ_i, $i=1,2,\ldots..,10l$.

Hence

$$\begin{aligned} &\mu_1 = \mu_2 = \cdots = \mu_l = 3 \\ &\mu_{l+1} = \mu_{l+2} = \cdots = \mu_{6l} = 1 \\ &\mu_{6l+1} = \mu_{6l+2} = \cdots = \mu_{10l} = -2 \end{aligned} \quad (1)$$

$$\mu_{6l+1} = \mu_{6l+2} = \ldots.. = \mu_{10l} = -2$$

Let the eigenvalues of G be,

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{10l+1} \quad (2)$$

Since B is the principal submatrix of the real symmetric matrix of A and by the interlacing Theorem 2.2.1, the eigenvalues sequence of (1) interlaces the eigenvalues sequence of (2).

Hence, we get

$$\begin{aligned} &\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \cdots \geq \mu_{l-1} \geq \\ &\lambda_l \geq \mu_l \geq \lambda_{l+1} \geq \mu_{l+1} \geq \lambda_{l+2} \ldots \geq \\ &\lambda_{6l} \geq \mu_{6l} \geq \lambda_{6l+1} \geq \mu_{6l+1} \geq \cdots \geq \\ &\mu_{10l} \geq \lambda_{10l} \end{aligned} \quad (3)$$

Using the values μ_i, $i=1,2,\ldots..,10l$, (3) reduces to (4).

$$\begin{aligned} &\lambda_1 \geq 3 \geq \lambda_2 \geq 3 \geq \cdots \geq \mu_{l-1} \geq \lambda_l \geq 3 \\ &\geq \lambda_{l+1} \geq 1 \geq \lambda_{l+2} \ldots \geq \lambda_{6l} \geq 1 \geq \lambda_{6l+1} \\ &\geq -2 \geq \cdots \geq -2 \geq \lambda_{10l} \end{aligned} \quad (4)$$

The inequality (4) yields the $10l$-3 eigenvalues of G as,

$$\begin{aligned} &\lambda_2 = \lambda_3 = \cdots = \lambda_l = 3 \\ &\lambda_{l+2} = \lambda_{l+3} = \cdots = \lambda_{6l} = 1 \\ &\lambda_{6l+2} = \lambda_{6l+3} = \cdots = \lambda_{10l} = -2 \end{aligned}$$

and bounds for the remaining four eigenvalues are,

$$\begin{aligned} &a = \lambda_1 \geq 3 \\ &b = \lambda_{l+1} \in (1,3) \\ &c = \lambda_{6l+1} \in (-2,1) \\ &d = \lambda_{10l+1} \leq -2 \end{aligned} \quad (5)$$

Finding the characteristic polynomial of G in Fig. 2, using a divisor graph.

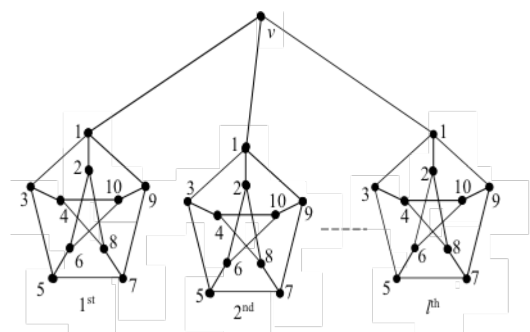

Fig. 2. Graph $G(v,lP,1)$

Divisor of G: Let the $V(G)$ vertex set of graph G of Fig 2, be partitioned into $4l+1$ sets, $X_v=\{v\}$, X_{-1}, X_2, ..., X_{4l}.

Vertices incident on v:

Let X_1, X_2, ... ,X_{4l} be the sets containing the 10 vertices of the 1st, 2nd..., lth copy of Petersen graph P of G-v respectively. Any one of the vertices of these l copies may be incident to vertex v because they are all of the same degree but to demonstrate meaningful construction and to follow the labeling in the proof, all vertices of

X_{4i-3} , $i=1,2,...,l$ incident to v.

Vertices, not incident on v:

Label the vertices of l copies of the Petersen graph of G-v as, 1,2,…,10. The vertices are not adjacent to v belong to the X-sets as shown below for $i=1,2,…,l$.

$$X_2 = X_6 = \cdots = X_{4i-2} = \{2,3,9\}$$
$$X_3 = X_7 = \cdots = X_{4i-1} = \{6,4,7\}$$
$$X_4 = X_8 = \cdots = X_{4i} = \{5,8,10\}$$

Before we draw the divisor graph F of graph G, we have the following assertions because of the above partition.

Observation:

The divisor graph is a multigraph.

i) Vertex v of a singleton set $X_v=\{v\}$ is adjacent to singleton vertex sets $X_{4i-3}=\{1\}$, $i=1,2,...,l$.

ii) Vertex {1} of X_{4i-3} , $i=1,2,...,l$ is adjacent to exactly one vertex of X_v (to vertex v).

iii) Vertex v and each vertex of X_{4l-2} ,X_{4l-1} ,X_{4l} are mutually adjacent to zero vertices.

iv) Each vertex of X_{4i-3} is adjacent to exactly 3 vertices of X_{4i-2} , $i=1,2,...,l$ and vertices of X_{4i-3} are not adjacent to the vertices of X_{4i-1} and X_{4i}, for $i=1,2,...,l$.

v) Each vertex of X_{4i-2} , $i=1,2,...,l$, is adjacent to one vertex of X_{4i-3} , X_{4i-1} and X_{4i} , for $i=1,2,...,l$.

vi) Each vertex of X_{4i-1} is adjacent to one vertex of X_{4i-2} and two vertices of X_{4i}, for $i=1,2,…,l$.

vii) Each vertex of X_{4i} is adjacent to one vertex of X_{4i-2} and two vertices of X_{4i-1} , for $i=1,2,…,l$.

viii) There is no vertex in the sets X_v , X_1,X_2 ,…,X_{4i}, $i=1,2,...,l$ is adjacent to itself (since no loop).

In the above sentences, vertex x is adjacent to y means that there is a (directed) arc going from x to y.

This partition, therefore, satisfies the concept of the divisor, and the corresponding divisor F is shown in Fig. 3, and graph F is not a digraph. The arrow on the edges shows the number of edges from X_i to X_j.

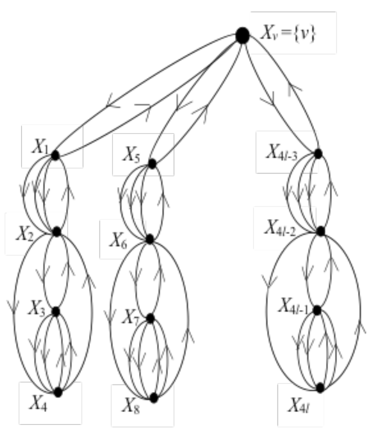

Fig. 3. Divisor Graph F of $G(v,lP,1)$

For the characteristic polynomial of divisor graph F, we first find |C-λI| and lastly multiply by $(-1)^{4l+1} = -1$, as(C-λI) is a $(4l+1)\times(4l+1)$ matrix with rows and columns labeled as X_v, X_1 , X_2 , X_3 , X_4 , X_5, X_6 , X_7 , X_8 ,…, X_{4l-3} , X_{4l-2} , X_{4l-1} , X_{4l} . The determinant of(C-λI) takes the form of the determinant (6).

$$|C-\lambda I| = \begin{vmatrix} -\lambda & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \dots & 1 & 0 & 0 & 0 \\ 1 & -\lambda & 3 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 1 & -\lambda & 1 & 1 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -\lambda & 2 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & -\lambda & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & -\lambda & 3 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -\lambda & 1 & 1 & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -\lambda & 2 & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & -\lambda & \dots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & -\lambda & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 1 & -\lambda & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 1 & -\lambda & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 1 & 2 & -\lambda \end{vmatrix} \quad (6)$$

To get the matrix in the block diagonal matrix form, we need to perform the following elementary operations.

Step1. Adding columns,

$$X_5, X_9, \ldots, X_{4l-3} \text{ to } X_1$$
$$X_6, X_{10}, \ldots, X_{4l-2} \text{ to } X_2$$
$$X_7, X_{11}, \ldots, X_{4l-1} \text{ to } X_3$$
$$X_8, X_{12}, \ldots, X_{4l} \text{ to } X_4$$

Step 2. Performing row operations,

$$X_{4l-3} - X_1, \quad i = 2 \text{ to } l$$
$$X_{4l-2} - X_2, \quad i = 2 \text{ to } l$$
$$X_{4l-1} - X_3, \quad i = 2 \text{ to } l$$
$$X_{4l} - X_4, \quad i = 2 \text{ to } l$$

Then so obtained determinant is in the block diagonal form can be factorized in the form using lemma 2.2.7.

$$|C - \lambda I| = \begin{vmatrix} -\lambda & l & 0 & 0 & 0 \\ 1 & -\lambda & 3 & 0 & 0 \\ 0 & 1 & -\lambda & 1 & 1 \\ 0 & 0 & 1 & -\lambda & 2 \\ 0 & 0 & 1 & 2 & -\lambda \end{vmatrix} \begin{vmatrix} -\lambda & 3 & 0 & 0 \\ 1 & -\lambda & 1 & 1 \\ 0 & 1 & -\lambda & 2 \\ 0 & 1 & 2 & -\lambda \end{vmatrix}^{l-1} \quad (7)$$

Simplifying (7) and multiplying by $(-1)^{4l+1} = -1$ on both the sides, we get the characteristic polynomial of the F as,

$$|\lambda I - C| = (\lambda^4 - 2\lambda^3 - (5+l)\lambda^2 + (6+2l)\lambda + 2l)(\lambda - 3)^{l-1}(\lambda + 2)^{2l-1}(\lambda - 1)^{l-1} \quad (8)$$

Let $f(\lambda) = \lambda^4 - 2\lambda^3 - (5+l)\lambda^2 + (6+2l)\lambda + 2l$ is the first factor of $|\lambda I - C|$. Now claim that, $f(\lambda) = 0$ has no solution as

$$\lambda = 3, \lambda = 1 \text{ and } \lambda = -2 \quad (9)$$

For, if possible, let $\lambda = 3$ be the root of $f(\lambda) = 0$ then $f(3) = 0 \xrightarrow{yields} -l = 0 \xrightarrow{yields} l = 0$

Contradiction to the fact that $l \geq 1$.

Similarly, we can show that $\lambda=1$ and $\lambda=-2$ are not the roots of $f(\lambda)=0$.

From equation (9), $f(\lambda)=0$ has no solution for, $\lambda=3, \lambda=1$, and $\lambda=-2$.

With this observation and by the Theorem 2.2.1, Theorem 2.2.5 with the equations (5) and (9), proof of the theorem. ■

Theorem 2: For vertex v and $l \geq 1$ the characteristic polynomial of $G(v, lC_6, 1)$ is

$$\emptyset(G(v, lC_6, 1):\lambda) = (\lambda^2 - 4)^{l-1}(\lambda^2 - 1)^{2l-1}\ (\lambda^5 - (5+l)\lambda^3 + (3l+4)\lambda);\ for\ l \geq 1$$

Proof: The proof can be achieved by having similar arguments as Theorem 1. Using the fact that, l copies of *C_6* of the graph *G-v*, having the *6l* eigenvalues on the whole as, l times *2, 2l* times *1, 2l* times -1, and *l* times -2.

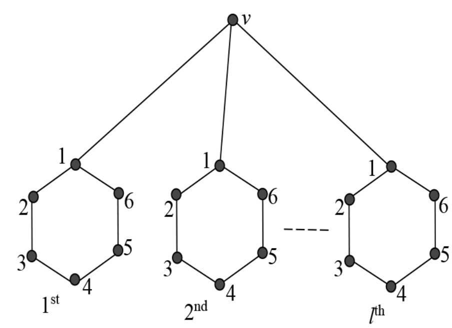

Fig. 4. Graph $G(v, lC_6, 1)$

Let the eigenvalues of G be $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_{6l+1}$ and

$$\lambda_2 = \lambda_3 = \cdots = \lambda_l = 2$$
$$\lambda_{l+2} = \lambda_{l+3} = \cdots = \lambda_{3l} = 1$$
$$\lambda_{3l+2} = \lambda_{3l+3} = \cdots = \lambda_{5l} = -1$$
$$\lambda_{5l+2} = \lambda_{5l+3} = \cdots = \lambda_{6l} = -2$$

and bounds for the remaining four eigenvalues are,

$$\begin{aligned} a &= \lambda_1 \geq 2 \\ b &= \lambda_{l+1} \in (1,2) \\ c &= \lambda_{3l+1} \in (-1,1) \\ d &= \lambda_{5l+1} \in (-2,-1) \\ e &= \lambda_{6l+1} \leq -2 \end{aligned} \quad (10)$$

Let $V(G)$ of Figure 4, be partitioned in $4l+1$ sets $X_v=\{v\}, X_1, X_2, \ldots\ldots, X_{4l}$.

Vertices incident on v:

Let $X_1, X_2, \ldots\ldots, X_{4l}$ be the sets containing the remaining 6 vertices of the 1st, 2nd,..., lth copy of C_6 of G-v respectively, such that all the vertices of X_{4i-3}, $i=1, 2, \ldots, l$ are incident on v.

Vertices not incident on v:
Label the vertices of l copies of C_6 of G-v as, $1,2,\ldots,6l$.
The vertices are not adjacent to v belong to the X-sets as shown below, for $i=1,2,\ldots,l$.

$$X_2 = X_6 = \cdots = X_{4i-2} = \{2,6\}$$
$$X_3 = X_7 = \cdots = X_{4i-1} = \{3,5\}$$
$$X_4 = X_8 = \cdots = X_{4i} = \{4\}$$

Now following this partition, the divisor graph F for $G(v,lC_6,1)$ is given in Figure 5,

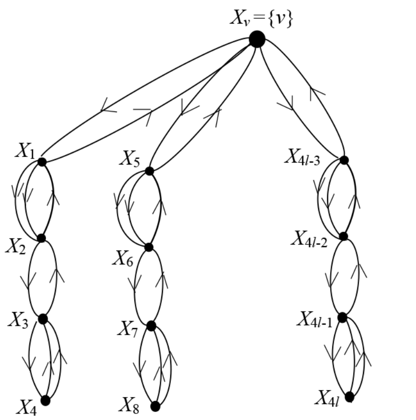

Fig. 5. Divisor graph F for $G(v,lC_6,1)$

We have the characteristic polynomial of F by evaluating the determinant,

$$|C-\lambda I| = \begin{vmatrix} -\lambda & l & 0 & 0 & 0 \\ 1 & -\lambda & 2 & 0 & 0 \\ 0 & 1 & -\lambda & 1 & 1 \\ 0 & 0 & 1 & -\lambda & 1 \\ 0 & 0 & 1 & 2 & -\lambda \end{vmatrix} \begin{vmatrix} -\lambda & 2 & 0 & 0 \\ 1 & -\lambda & 1 & 0 \\ 0 & 1 & -\lambda & 1 \\ 0 & 0 & 2 & -\lambda \end{vmatrix}^{l-1} \qquad (11)$$

Simplifying (11) and multiplying by $(-1)^{4l+1}=-1$ on both the sides, we get the characteristic polynomial of divisor graph F as,

$$|\lambda I - C| = (\lambda^5 - (5+l)\lambda^3 + (4+3l)\lambda)$$
$$(\lambda^2-4)^{l-1}(\lambda^2-1)^{l-1} \qquad (12)$$

Let $f(\lambda)=\lambda^5-(5+l)\lambda^3+(4+3l)\lambda$ which is the first factor of $|\lambda I-C|$. Now claim that, $f(\lambda)=0$ has no solution for,

$$\lambda = \pm 1, \lambda = \pm 2 \qquad (13)$$

For, if possible, let $\lambda = 2$ be a root of $f(\lambda)=0$ then $f(2)=0 \xrightarrow{yields} -2l=0 \xrightarrow{yields} l=0$
Contradiction to the fact that $l \geq 1$.
Similarly, $\lambda = \pm 1$ and $\lambda = -2$ are not the roots of $f(\lambda)=0$.

From equation (13), $f(\lambda)=0$ has no solution for, $\lambda=\pm 1$ and $\lambda=\pm 2$.
With this observation and by the Theorem 2.2.1, Theorem 2.2.5, with the equations (10) and (13) the proof of the theorem 2. ■
Theorem 3: For vertex v and $l\geq 1$ the characteristic polynomial of $G(v,lK_{n,n},1)$ is given in the following equation,

$$\emptyset(G(v,lK_{n,n},1):\lambda) = (\lambda+n)^{l-1}(\lambda-n)^{l-1}\lambda^{2nl-2l-1}(\lambda^4-(n^2+l)\lambda^2 + (n^2l-nl)); \qquad for\ l\geq 1, n\geq 3$$

Proof: Following similar arguments of Theorem 1 and Theorem 2, G and G-v are graphs of order $2nl$+1 and $2nl$ respectively. The eigenvalues of G-v are l times n, $(2nl-2l)$ times 0, and l times $-n$. The eigenvalues of G-v are $2nl$.

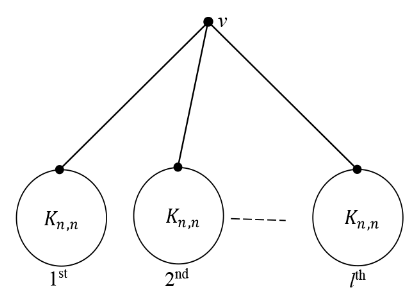

Fig. 6. Graph $G(v,lK_{n,n},1)$

Let the eigenvalues of G be $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_{2nl+1}$ and

$$\lambda_2 = \lambda_3 = \cdots = \lambda_l = n$$
$$\lambda_{l+2} = \lambda_{l+3} = \cdots = \lambda_{2nl-l} = 0$$
$$\lambda_{2nl-l+2} = \lambda_{2nl-l+3} = \cdots = \lambda_{2nl} = -n$$

and bounds for the remaining four eigenvalues are,

$$\begin{aligned} a &= \lambda_1 \geq n \\ b &= \lambda_{l+1} \in (0, n) \\ c &= \lambda_{2nl-l+1} \in (0, -1) \\ d &= \lambda_{2nl+1} \leq -n \end{aligned} \qquad (14)$$

The divisor graph F of Fig. 6. is shown in Fig. 7. One can have clear information on the partition of the vertex set and draw the divisor graph.

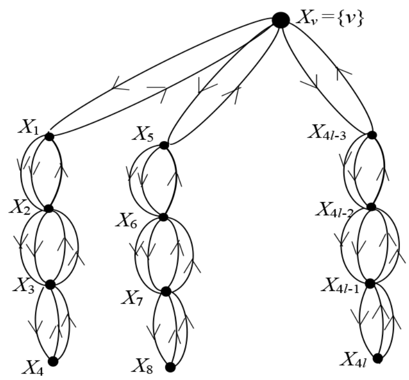

Fig. 7. Divisor Graph *F* of $G(v, lK_{n,n}, 1)$

Now, the characteristic polynomial of *F*, det $(C-\lambda I)$ and multiplying it by $(-1)^{4l+1}=-1$ as $|C-\lambda I|$ is a $(4l+1)\times(4l+1)$ matrix. Simplifying (15) and multiplying $(-1)^{4l+1}=-1$ on both the sides, we get the characteristic polynomial of the divisor graph *F* as,

$$|C-\lambda I| = \begin{vmatrix} -\lambda & l & 0 & 0 & 0 \\ 1 & -\lambda & n-1 & 0 & 1 \\ 0 & 1 & -\lambda & n-1 & 0 \\ 0 & 0 & n-1 & -\lambda & 1 \\ 0 & 1 & 0 & n-1 & -\lambda \end{vmatrix} \begin{vmatrix} -\lambda & n-1 & 0 & 1 \\ 1 & -\lambda & n-1 & 0 \\ 0 & n-1 & -\lambda & 1 \\ 1 & 0 & n-1 & -\lambda \end{vmatrix}^{l-1} \qquad (15)$$

$$|\lambda I - C| = \left(\lambda^4 - (n^2+l)\lambda^2 + (n^2 l - nl)\right)(\lambda + n)^{l-1}(\lambda-n)^{l-1}\lambda^{2l-1}(\lambda+n)^{l-1} \qquad (16)$$

Let $f(\lambda)=\lambda^4-(n^2+l)\ \lambda^2+(n^2l-nl)$ which is the first factor of $|\lambda I-C|$. Now claim that, $f(\lambda)=0$ has no solution as

$$\lambda = \pm n \text{ and } \lambda = 0 \qquad (17)$$

For, if possible, let $\lambda=n$ be a root of $f(\lambda)=0$ then $f(n)=0 \xrightarrow{yields} -nl=0 \xrightarrow{yields} l=0 \text{ or } n=0$ Contradiction to the fact that $l\geq 1$ and $n\geq 3$. Similarly, we can show that $\lambda=-n$ and $\lambda=0$ are not the roots of $f(\lambda)=0$.

From equation (17) $f(\lambda)=0$ has no solution for, $\lambda=\pm n$, and $\lambda=0$.

With this observation and by the Theorem 2.2.1, Theorem 2.2.5, with the equations (14) and (17) the proof of the theorem 3. ■

Theorem 4: For vertex *v* and $l\geq 1$ the characteristic polynomial of $G(v,lH,1)$ is as given in the equation below,

$$\begin{aligned} \emptyset(G(v,lH,1):\lambda) &= (\lambda+3)^{l-1}(\lambda-3)^{l-1}(\lambda^2-1)^{3l-1}(\lambda^5 \\ &\quad -(10+l)\lambda^3+(9+7l)\lambda)\,; \quad for\ l\geq 1 \end{aligned}$$

Where H is the graph obtained from $K_{4,4}$ by deleting 1- factor.

Proof: From similar arguments of Theorem 1, 2, and 3, the graph *G* and *G-v* are graphs of order $8l+1$ and $8l$ respectively. The eigenvalues of *G-v* are *l* times *3, 3l* times *1, 3l* times -1, and *l* times -3. The eigenvalues of *G-v* are *8l.*

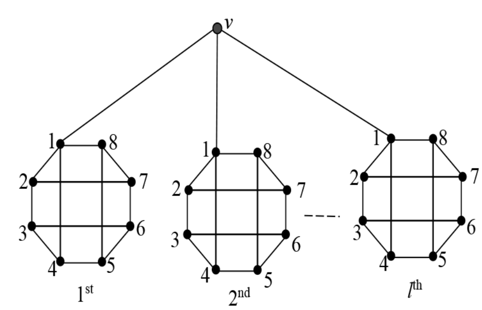

Fig. 8. Graph $G(v,lH,1)$

Let the eigenvalues of *G* be $\lambda_1\geq\lambda_2\geq\ldots\geq\lambda_{8l+1}$ and

$$\begin{aligned}
\lambda_2 &= \lambda_3 = \cdots = \lambda_l = 3\\
\lambda_{l+2} &= \lambda_{l+3} = \cdots = \lambda_{4l} = 1\\
\lambda_{4l+2} &= \lambda_{4l+3} = \cdots = \lambda_{7l} = -1\\
\lambda_{7l+2} &= \lambda_{7l+3} = \cdots = \lambda_{8l} = -3
\end{aligned}$$

and bounds for the remaining five eigenvalues are,

$$\begin{aligned}
a &= \lambda_1 \geq 3\\
b &= \lambda_{l+1} \in (1,3)\\
c &= \lambda_{4l+1} \in (-1,1)\\
d &= \lambda_{7l+1} \in (-3,-1)\\
e &= \lambda_{8l+1} \leq -3
\end{aligned} \tag{18}$$

The divisor graph *F* of Fig. 8, is shown in Fig. 9.

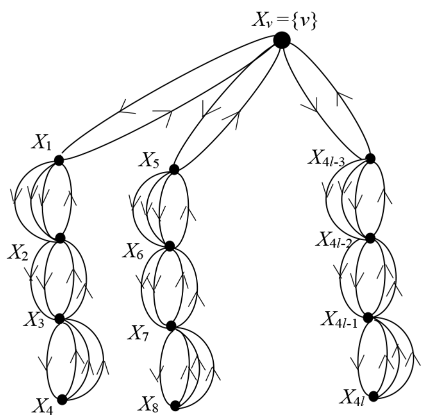

Fig. 9. Divisor Graph *F* of *G(v,lH,1)*

Now, we find the characteristic polynomial of F as det(C- λI) and multiply it by $(-1)^{4l+1}=-1$ as $|C-\lambda I|$ is a $(4l+1)\times(4l+1)$ matrix.

$$|C-\lambda I| = \begin{vmatrix} -\lambda & l & 0 & 0 & 0\\ 1 & -\lambda & 3 & 0 & 0\\ 0 & 1 & -\lambda & 2 & 0\\ 0 & 0 & 2 & -\lambda & 1\\ 0 & 0 & 0 & 3 & -\lambda \end{vmatrix} \begin{vmatrix} -\lambda & 3 & 0 & 0\\ 1 & -\lambda & 2 & 0\\ 0 & 2 & -\lambda & 1\\ 0 & 0 & 3 & -\lambda \end{vmatrix}^{l-1} \tag{19}$$

Simplifying (19) and multiplying by −1 on both the sides, we get the characteristic polynomial of the divisor graph *F* as,

$$|\lambda I - C| = (\lambda^5 - (10+l)\lambda^3 + (9+7l)\lambda)\,(\lambda+3)^{l-1}(\lambda-3)^{l-1}(\lambda^2-1)^{3l-1} \tag{20}$$

Let $f(\lambda)=\lambda^5-(10+l)\lambda^3+(9+7l)\lambda$ which is the first factor of $|\lambda I-C|$.

Now we claim that, $f(\lambda)=0$ has no solution as $\lambda = \pm 3$ and $\lambda = \pm 1$ (21)

For, if possible, let $\lambda = 3$ be a root of $f(\lambda)=0$ then $f(3)=0 \xrightarrow{yields} -6l=0 \xrightarrow{yields} l=0$.

Contradiction to the fact that $l\geq 1$.

Similarly, we can show that $\lambda = -3$ and $\lambda = \pm 1$ are not the roots of $f(\lambda)=0$.

From equation (21) $f(\lambda)=0$ has no solution for, $\lambda = \pm 3$ and $\lambda = \pm 1$.

With this observation and by the Theorem 2.2.1, Theorem 2.2.5, with the equations (18) and (21) the proof of the theorem. ■

3. Minimal graphs

In this section, we prove that the class of graphs *G(v,lP,1)*, $G(v,lC_6,1)$, $G(v,lK_{—n,n},1)$ and *G(v,lH,1)* exhibits minimality conditions.

Proposition 1: For l>0 the class of graphs G(v,lP,1) of theorem 1 is minimal.

Proof: For l>0 the diameter,

d = diam(G(v,lP,1)) is 6. From the characteristic polynomial of *G(v,lP,1)* the spectra of this graph,

$$Spect(G(v,lP,1)) = \begin{Bmatrix} 3 & 1 & -2 & a & b & c & d\\ l-1 & 5l-1 & 4l-1 & 1 & 1 & 1 & 1 \end{Bmatrix}$$

From equation(9), eigenvalues a, b, c, and d are other than 3,1,-2.

Hence the number of distinct eigenvalues $= d+1=7$. By the definition 2.2.6, hence proved. ■

Proposition 2: For l>0 the class of graphs *G(v,lC_6,1)* of theorem 2 is minimal.

Proof: For l>0 the diameter,

$d = diam(G(v,lC_6,1))$ is 8.

From the characteristic polynomial of $G(v,lC_6,1)$ the spectra of this graph,

$$Spect(G(v,lC_6,1)) = \begin{Bmatrix} 2 & 1 & -2 & -1 & a & b & c & d & e\\ l-1 & 2l-1 & l-1 & 2l-1 & 1 & 1 & 1 & 1 & 1 \end{Bmatrix}$$

From equation(13), the eigenvalues a, b, c, and d are other than 2,1,-2, and -1.

Hence the number of distinct eigenvalues = d+1=9. By the definition 2.2.6, hence proved. ■

Proposition 3: For l>0 the class of graphs $G(v,lK_{n,n},1)$ of theorem 3 is minimal.

Proof: For l>0 the diameter,

$d = diam(G(v,lK_{n,n},1))$ is 6.

From the characteristic polynomial of $G(v,lK_{n,n},1)$ the spectra of this graph:

$$Spect\left(G\left(v, lK_{n,n}, 1\right)\right) = \begin{Bmatrix} n & 0 & -n & a & b & c & d \\ l-1 & 2nl-2l-1 & l-1 & 1 & 1 & 1 & 1 \end{Bmatrix}$$

From equation(17), the eigenvalues a,b,c, and d are other than n,0 and-n.

Hence the number of distinct eigenvalues = d+1=7. By the definition 2.2.6, hence proved. ■

Proposition 4: For l>0 the class of graphs $G(v,lH,1)$ of theorem 4 is minimal.

Proof: For l>0 the diameter,

$d = diam(G(v,lH,1))$ is 8.

From the characteristic polynomial of $G(v,lC_6,1)$ the spectra of this graph,

$$Spect(G(v, lH, 1)) = \begin{Bmatrix} 3 & 1 & -1 & -3 & a & b & c & d & e \\ l-1 & 3l-1 & 3l-1 & l-1 & 1 & 1 & 1 & 1 & 1 \end{Bmatrix}$$

From equation(21), the eigenvalues a, b, c, and d are other than 3,1,-1, and -3.

Hence the number of distinct eigenvalues = d+1=9. By the definition 2.2.6, hence proved. ■

Observation: All the cycles of even lengths, strongly regular graphs are minimal graphs following this construction.

Conclusion

The consequences of this paper examine the construction of a class of non-regular minimal graphs with large matrices. Along with the investigation of their characteristic polynomial and spectra by using the divisor graph method. *A* cycle of length six has been proven to be minimal, as have all cycles of even length on observation. Furthermore, after a similar construction, all strongly regular graphs are also minimal. Although it is difficult to characterize regular or nonregular minimal graphs. We want to use graphs with known spectra to uncover potential classes of nonregular minimal graphs in the future.

Conflict of interest

The authors declare no conflict of interest.

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